Comparing tail variabilities of risks by means of the excess wealth order

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Abstract

There is a growing interest in the actuarial community to employ certain tail conditional characteristics as measures of risk, which are informative about the variability of the losses beyond the value-at-risk (one example is the tail conditional variance, introduced by Furman and Landsman, 2006). However, comparisons of tail risks based on different measures may not always be consistent. In addition, conclusions based on these conditional characteristics depend on the choice of the tail probability p, so different p's also may produce contradictory conclusions. In this note, we suggest to compare tail variability of risks by means of the excess wealth order, which makes judgements only if large classes of tail conditional characteristics imply the same conclusion, independently of the choice of p.

MSC: IM30

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1 Introduction

It is well-known that the value-at-risk is insensitive to the severity of losses in the tail of the distributions. Consequently, there is growing interest in the use of certain tail conditional characteristics as measures of risk, which are informative about the magnitude and variability of the losses beyond the value-at-risk. If X is a loss random variable, or risk, with distribution function F and corresponding

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quantile function F^{-1} defined by $F^{-1}(p) = \inf\{x : F(x) \geq p\}$, $0 \leq p \leq 1$, an example is the tail conditional expectation defined by

$$\mathbf{TCE}_p(X) = E\left(X \,\middle|\, X > F^{-1}(p)\right),$$

which represents the conditional expected loss given that the loss exceeds the value $F^{-1}(p)$. Given two risks X and Y, once the level p is fixed, the tail conditional expectation can be used to compare them and to determine which one is more dangerous. If we prefer to make comparisons not depending on the choice of p, we can consider robust comparisons based on the condition

$$\mathbf{TCE}_p(X) \leq \mathbf{TCE}_p(Y) \text{ for all } p \in (0,1) \tag{1}$$

generating a partial order in the set of random variables (for this topic, stochastic orderings, see Müller and Stoyan (2002), Denuit et al. (2005) and Shaked and Shanthikumar (2007)). Thus, for example, if X and Y are continuous, (1) is equivalent to say that X is smaller than Y in the stop-loss order, also called the increasing convex order (see, e.g., Levy (2006, Section 4)).

Recently, a number of authors, including Valdez (2005), Furman and Landsman (2006a, 2006b), Rachev et al. (2008, Sec. 7.9.1), Furman and Zitikis (2008), Bernard and Tian (2009) and Landsman (2009), have considered characteristics of conditional loss distributions for measuring the variability of the risk along the tail of its distribution. However, as it is well recognized, particular characteristics provide only limited information. In addition to this, comparisons of risks based on tail conditional characteristics depend on the choice of the tail probability p, so different p's may produce contradictory conclusions. The concept of dilation, as introduced by Hickey (1986), provides a more robust approach to the comparison of tail risks in terms of the variability. We can consider classes of measures of the form

$$H_{\varphi,p}(X) = E\left(\varphi\left(X - E\left(X_p\right)\right) \,\middle|\, X > F^{-1}(p)\right) \tag{2}$$

where $X_p = \mathcal{L}\left(X | X > F^{-1}(p)\right)$, $0 < p < 1$ and φ is a convex real function such that (2) exists. Clearly, $H_{\varphi,p}$ generalizes the use of the tail conditional variance (Furman and Landsman, 2006a) obtained when $\varphi(t) = t^2$ in (2), to compare tail risks in terms of dispersion. Then we can say that X is smaller than Y in some stochastic sense if

$$H_{\varphi,p}(X) \leq H_{\varphi,p}(Y), \text{ for all } p \in (0,1), \text{ for all } \varphi \text{ convex.} \tag{3}$$

The main purpose of this note is to show that (3) characterizes a well-known stochastic order among X and Y: the excess wealth order, also called the right spread order, whose definition is recalled (Fernández-Ponce et al., 1998; Shaked and Shanthikumar, 1998).

Definition 1 *Let X and Y be two random variables with distribution functions F and G, respectively. Then, we say that X is smaller than Y in the excess wealth order (denoted by $X \leq_{ew} Y$) if*

$$E\left[\left(X - F^{-1}(p)\right)^+\right] \leq E\left[\left(Y - G^{-1}(p)\right)^+\right], \textit{ for all } p \in (0,1).$$

In actuarial literature, the function $E\left[\left(X-F^{-1}(p)\right)^{+}\right]$, defined for $p \in (0,1)$, is called the expected shortfall at level p (Dhaene et al., 2006) and represents the expected shortfall of the portfolio with loss X and solvency capital requirement $F^{-1}(p)$. Therefore, the excess wealth order is a natural way of comparing the riskiness of two probability distributions. Denuit and Vermandele (1999), Chateauneuf et al. (2004), Hu et al. (2006) and Sordo (2008, 2009) provide some actuarial applications of this order. It is worth to note that whereas Sordo (2008) characterizes the excess wealth order by the spread of a risk throughout its distribution, here we rather focus on the tail risk.

Throughout this paper, increasing means non-decreasing.

2 The characterization

We require the following result from Ramos and Sordo (2003, Theorem 2.1).

Theorem 2 *Let X and Y be two random variables with respective distribution functions F and G and finite means μ_X and μ_Y, respectively. Then,*

$$E\left[\varphi\left(X-\mu_X\right)\right] \leq E\left[\varphi\left(Y-\mu_Y\right)\right] \text{ for all convex } \varphi \tag{4}$$

if, and only if,

$$\int_u^1 F^{-1}(t)-E(X)\,dt \leq \int_u^1 \left[G^{-1}(t)-E(Y)\right] dt,\ \forall u \in (0,1).$$

We also need the following characterization of the excess wealth order (see Section 3.C.1 in Shaked and Shanthikumar, 2007).

Theorem 3 *Let X and Y be two random variables with respective distribution functions F and G. Then, $X \leq_{ew} Y$ if, and only if,*

$$\frac{1}{1-a}\int_a^1 \left[G^{-1}(t)-F^{-1}(t)\right] dt \text{ is increasing in } a \in (0,1).$$

Now we can prove the following result.

Theorem 4 *Let X and Y be two continuous random variables with respective distribution functions F and G. Then, $X \leq_{ew} Y$ if and only if*

$$H_{\varphi,p}(X) \leq H_{\varphi,p}(Y),\ \text{for all } p \in (0,1),\ \text{for all } \varphi \text{ convex.} \tag{5}$$

Proof. For each $p \in (0,1)$, let F_p be the distribution function of X_p, given by

$$F_p(x)=\begin{cases} 0 & \text{if } x \leq F^{-1}(p) \\ \frac{F(x)-p}{1-p} & \text{if } x > F^{-1}(p) \end{cases}$$

and denote by F_p^{-1} the corresponding quantile function, given by

$$F_p^{-1}(t)=F^{-1}\left[p+(1-p)t\right],\ \forall t \in (0,1).$$

Similarly, let G_p be the distribution function of Y_p and denote by G_p^{-1} the corresponding quantile function. From Theorem 2 it follows that (5) is equivalent to

$$\int_u^1 \left[F_p^{-1}(t) - E\left(X_p\right)\right] dt \leq \int_u^1 \left[G_p^{-1}(t) - E\left(Y_p\right)\right] dt,\ \forall u \in (0,1), \forall p \in (0,1). \tag{6}$$

Taking into account that

$$E\left(X_p\right) = \frac{\int_p^1 F^{-1}(x)dx}{1-p},\ p \in (0,1),$$

we have that (6) is equivalent to

$$\int_u^1 \left(F^{-1}\left[p + (1-p)t\right] - \frac{\int_p^1 F^{-1}(x)dx}{1-p}\right) dt \leq \tag{7}$$

$$\leq \int_u^1 \left(G^{-1}\left[p + (1-p)t\right] - \frac{\int_p^1 G^{-1}(x)dx}{1-p}\right) dt,\ u \in (0,1), p \in (0,1).$$

A change of variable shows that (7) is the same as the condition

$$\frac{\int_{p+(1-p)u}^1 \left[G^{-1}(t) - F^{-1}(t)\right] dt}{(1-u)(1-p)} \geq \tag{8}$$

$$\geq \frac{\int_p^1 \left[G^{-1}(t) - F^{-1}(t)\right] dt}{1-p},\ u \in (0,1), p \in (0,1).$$

Condition (8) is equivalent to

$$\frac{1}{1-a}\int_a^1 \left[G^{-1}(t) - F^{-1}(t)\right] dt \text{ is increasing in } a \in (0,1)$$

which means $X \leq_{ew} Y$ from Theorem 3. ■

The tail conditional variance (Furman and Landsman, 2006a), given by

$$\mathbf{TV}_p(X) = Var\left(X \left| X > F^{-1}(p)\right.\right) = E\left(\left(X - E\left(X_p\right)\right)^2 \left| X > F^{-1}(p)\right.\right), \tag{9}$$

with $0 < p < 1$, is a measure of the form (2) with $\varphi(t) = t^2$. The following corollary states that (9) is consistent with respect to the excess wealth order.

Corollary 5 *Let X and Y be two continuous random variables with respective distribution functions F and G. Then, $X \leq_{ew} Y$ implies*

$$Var\left(X \left| X > F^{-1}(p)\right.\right) \leq Var\left(Y \left| Y > G^{-1}(p)\right.\right) \text{ for all } p \in (0,1).$$

In order to state the next result, we consider a large class of variability measures of the form

$$I_\omega(X) = \int_0^1 \omega(p) \left[F^{-1}(p) - \mu_X\right] dp \tag{10}$$

where $\omega : [0,1] \longrightarrow [0,1]$. Functionals of the form (10) can be found, for example, in Table 7.8 of Nygard and Sandström (1981). Ramos and Sordo (2003) proved that (4) holds if, and only if,

$$I_\omega(X) \leq I_\omega(Y) \text{ for all increasing } \omega.$$

The following result easily follows from this observation and Theorem 4.

Corollary 6 *Let X and Y be two continuous random variables with respective distribution functions F and G. Then, $X \leq_{ew} Y$ if and only if*

$$I_\omega\left(X \left| X > F^{-1}(p)\right.\right) \leq I_\omega\left(Y \left| Y > G^{-1}(p)\right.\right) \;\; \forall\, p \in (0,1), \forall \text{ increasing } \omega. \tag{11}$$

The choice $\omega(t) = 4t$ in the previous corollary shows that the tail Gini's mean difference[1], given by

$$\mathbf{Gmd}_p(X) = Gmd\left(X \left| X > F^{-1}(p)\right.\right),\; 0 < p < 1, \tag{12}$$

where

$$Gmd(X) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |y - x| \, dF(x) dF(y),$$

is also consistent with the excess wealth order. It is worth to note that the Gini's mean difference cannot be written in the form (2) with Φ convex, as shown by Newbery (1970).

3 Further results

The information given by the tail standard deviation (equals to $(\mathbf{TV}_p(X))^{1/2}$) and the conditional expectation is combined in the tail standard deviation premium (**TSD**), defined by Furman and Landsman (2006a, 2006b) as

$$\mathbf{TSD}_p(X) = \mathbf{TCE}_p(X) + \alpha \left(\mathbf{TV}_p(X)\right)^{1/2} \tag{13}$$

where $0 < p < 1$ and α is some non-negative constant. This premium principle is particularly useful when the variability along the right-tail is crucial for decision makers. The popular standard deviation premium calculation principle **SD** (see Bühlmann, 1970, Chapter 4), given by

$$\mathbf{SD}(X) = E(X) + \alpha \left(Var(X)\right)^{1/2}$$

[1]Tail Gini type measures have been considered before in portfolio optimization problems by Ogryczak and Ruszczynski (2002a, 2002b).

is a particular case of $\mathbf{TSD}_p(X)$, which can be seen by letting $p \longrightarrow 0$ in (13). In addition to some other properties, such as non-negative loading, translation invariance and positive homogeneity, (13) satisfies the next tail parity property (Furman and Landsman, 2006a): if some $p \in (0,1)$ exists such that $F(t) = G(t)$ for all $t \geq F^{-1}(p)$, then $\mathbf{TSD}_p(X) = \mathbf{TSD}_p(Y)$, i.e., the tail standard deviation premium depends only on the tail of the distribution. This property is specially useful in the case of reinsurance contracts and policies involving deductibles. The next result is a sort of reciprocal, under excess wealth order, of this property. We can think of it as follows: under the excess wealth ordering, the tail of the distribution depends only of the tail standard deviation premium.

Theorem 7 *Let X and Y be two continuous random variables with respective distribution functions F and G such that $X \leq_{ew} Y$. If there exists $p \in (0,1)$ such that $\mathbf{TCE}_p(X) = \mathbf{TCE}_p(Y)$ and $\mathbf{TV}_p(X) = \mathbf{TV}_p(Y)$, then $X_p = \left(X \,\middle|\, X > F^{-1}(p)\right)$ and $Y_p = \left(Y \,\middle|\, Y > G^{-1}(p)\right)$ have the same distribution.*

Proof. If $X \leq_{ew} Y$ then (5) holds or, equivalently,

$$(X_p - E(X_p)) \leq_{cx} (Y_p - E(Y_p)), \text{for all } p \in (0,1), \tag{14}$$

where $\leq_{cx}$ denotes the well-known convex order (see Section 3.A in Shaked and Shanthikumar, 2007). By assumption, there exists $p_0 \in (0,1)$ such that

$$E(X_{p_0}) = E(Y_{p_0}) \tag{15}$$

and

$$E(X_{p_0} - E(X_{p_0}))^2 = E(Y_{p_0} - E(Y_{p_0}))^2. \tag{16}$$

From (14), (16) and Corollary 2.3 of Bhattachajee and Bhattacharya (2000) it follows that

$$X_{p_0} - E(X_{p_0}) \equiv_{st} Y_{p_0} - E(Y_{p_0}),$$

(where "$\equiv_{st}$" denotes equality in distribution) which, taking into account (15), implies that X_p and Y_p have the same distribution. ■

In particular, by letting $p \longrightarrow 0$ in the previous theorem, we see that two random variables with the same expectation and the same variance cannot be ordered via excess wealth order unless they have the same distribution.

A sufficient condition for X and Y to be ordered in the excess wealth order is the well-known dispersive order (Bickel and Lehmann, 1979), whose definition is recalled here.

Theorem 8 *Let X and Y be two random variables with respective distribution functions F and G. Then, X is said to be smaller than Y in the dispersive order (denoted by $X \leq_{disp} Y$) if*

$$F^{-1}(p) - F^{-1}(q) \leq G^{-1}(p) - G^{-1}(q), \text{ for all } 0 < q < p < 1.$$

Explicit expressions for tail variances and other tail conditional characteristics are not always available. However, many families of well-known distributions are ordered in the dispersive order according to the value of their parameters. Therefore, using that $X \leq_{disp} Y$ implies $X \leq_{ew} Y$ and Theorem 4, we can, for these families, compare tail risks in terms of large classes of measures, without needing their explicit expressions. The following simple example illustrate this application.

Example 9 *Let* $X \sim N(\mu_1, \sigma_1)$ *and* $Y \sim N(\mu_2, \sigma_2)$ *be two Normal distributions. It is well-known that* $\sigma_1 < \sigma_2$ *implies* $X \leq_{disp} Y$ *and so, in view of Theorem 4, we have that (5) and (11) hold. In particular,* $\mathbf{TV}_p(X) \leq \mathbf{TV}_p(Y)$ *and* $\mathbf{Gmd}_p(X) \leq \mathbf{Gmd}_p(Y)$ *for all* $p \in (0, 1)$.

More examples of dispersive-ordered families of distributions can be found in Saunders and Moran (1978), Lewis and Thomson (1981), Shaked (1982), Hickey (1986) and Rojo and He (1991), among others.

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